On the optimal dividend problem for a spectrally positive Lévy process

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Abstract

In this paper we study the optimal dividend problem for a company whose surplus process evolves as a spectrally positive Lévy process. This model including the dual model of the classical risk model and the dual model with diffusion as special cases. We assume that dividends are paid to the shareholders according to admissible strategy whose dividend rate is bounded by a constant. The objective is to find a dividend policy so as to maximize the expected discounted value of dividends which are paid to the shareholders until the company is ruined. We show that the optimal dividend strategy is formed by a threshold strategy.

Key Words: Threshold strategy, Dual model, Optimal dividend strategy, Scale functions, Spectrally positive Lévy process, Stochastic control.

1 INTRODUCTION

Recently, dividend optimization problems for financial and insurance corporations have attracted extensive attention. How should corporation pay dividends to its shareholders? A possible goal is that the company tries to maximize the expectation of the discounted dividends until possible ruin of the company. In recent years, quite a few interesting papers deal with the optimal dividend problem in the dual model. For example, Avanzi et al. (2007) consider the model which is dual to the classical risk mode, the authors studied how the expectation of the discounted dividends until ruin can be calculated when gain distribution has a exponential distribution or mixtures of exponential distributions and shows how the exact value of the optimal dividend barrier can be determined, if the jump distribution is known. Avanzi and Gerber (2008) examined the same problem for the dual model perturbed by diffusion. Moreover, they pointed out that "the optimal dividend strategy in the dual model is a barrier strategy. A direct proof that optimal strategy is a barrier strategy is of some interest but has not been given to our knowledge; the proof in Bayraktar and Egami (2008) is for exponential gains only." Yao et al. (2010) considered the optimal problem with dividend payments and issuance of equity in a dual risk model without a diffusion, assuming proportional transaction costs, they found optimal strategy which maximizes the expected present value of the dividends payout minus the discounted costs of issuing new equity before ruin. In addition, for exponentially distributed jump sizes, closed form solutions are obtained. Dai, et al. (2010, 2011) considered the same problem as in Yao et al. (2010) for a dual risk model with a diffusion with bounded gains and exponential gains, respectively. Avanzi, Shen and Wong (2011) determined an explicit form for the value function in the dual model with diffusion when the gains distribution is a mixture of exponentials. They showed that a barrier dividend strategy is also optimal and conjectured that the optimal dividend strategy in the dual model with diffusion should be the barrier strategy, regardless of the gains distribution. Until recently, Bayraktar, Kyprianou and Yamazaki (2013) using the fluctuation theory of spectrally positive Lévy processes, show the optimality of barrier strategies for all such Lévy processes.

Barrier strategies often serve as candidates for the optimal strategy when the dividend rate is unrestricted. However, if a barrier strategy is applied, ultimate ruin of the company is certain. In many circumstances this is not desirable. This consideration leads us to impose restriction on the dividend stream. Ng (2009) considered the dual of the compound Poisson model under a threshold dividend strategy and derived a set of two integro-differential equations satisfied by the expected total discounted dividends until ruin and showed how the equations can be solved by using only one of the two integro-differential equations. The cases where profits follow an exponential or a mixture of exponential distributions are then solved and the discussion for the case of a general profit distribution follows by the use of Laplace transforms. He illustrated how the optimal threshold level that maximizes the expected total discounted dividends until ruin can be obtained. In this paper we provide a uniform mathematical framework to analyze the optimal control problem with dividends payout for a general spectrally positive Lévy process. We assume that dividends are paid to the shareholders according to admissible strategies whose dividend rate is bounded by a constant. Under this additional constraint, we show that the optimal dividend strategy is formed by a threshold strategy.

The rest of the paper is organized as follows. Section 2 presents the model and formulates the dividend optimization problem. Section 3 discusses the threshold strategies. Explicit expressions for the expected discounted value of dividend payments are obtained, and in Section 4 we give the main results, it is shown that the optimal dividend strategy is formed by a threshold strategy with parameters b^* and α under the constraint that only dividend strategies with dividend rate bounded by α are admissible.

2 The model and the optimization problem

Let $X = \{X_t\}_{t\geq 0}$ be a spectrally positive Lévy process with non-monotone paths on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is generated by the process X and satisfies the usual conditions. The Lévy triplet of X is given by (c, σ, Π), where $c > 0$,

$\sigma \geq 0$ and Π is a measure on $(0,\infty)$ satisfying

$$\int_0^\infty (1 \wedge x^2)\Pi(dx) < \infty.$$

Denote by P_x for the law of X when $X_0 = x$. Let E_x be the expectation associated with P_x. For short, we write P and E when $X_0 = 0$. The Laplace exponent of X is given by

$$\Psi(\theta) = \frac{1}{t}\log Ee^{-\theta X_t} = c\theta + \frac{1}{2}\sigma^2\theta^2 + \int_0^\infty (e^{-\theta x} - 1 + \theta x \mathbf{1}_{\{|x|<1\}})\Pi(dx), \tag{2.1}$$

where $\mathbf{1}_A$ is the indicator function of a set A. In the sequel, we assume that $-\Psi'(0+) = \mathbb{E}(X_1) > 0$ which implies the process X drifts to $+\infty$. It is well known that if $\int_1^\infty y\Pi(dy) < \infty$, then $\mathbb{E}(X_1) < \infty$, and $\mathbb{E}(X_1) = -c + \int_1^\infty y\Pi(dy)$. Note that X has paths of bounded variation if and only if

$$\sigma = 0 \text{ and } \int_0^\infty (1 \wedge x)\Pi(dx) < \infty.$$

In this case, the Laplace exponent (2.1) can be written as

$$\Psi(\theta) = c_0\theta + \frac{1}{2}\sigma^2\theta^2 + \int_0^\infty (e^{-\theta x} - 1)\Pi(dx), \tag{2.2}$$

with $c_0 = c + \int_0^1 x\Pi(dx)$ the so-called drift of X.

For an arbitrary spectrally positive Lévy process, the Laplace exponent Ψ is strictly convex and $\lim_{\theta\to\infty}\Psi(\theta) = \infty$. Thus there exists a function $\Phi : [0,\infty) \to [0,\infty)$ defined by $\Phi(q) = \sup\{\theta \geq 0 | \Psi(\theta) = q\}$ such that $\Psi(\Phi(q)) = q,\ q \geq 0$.

For more details on spectrally positive Lévy processes, the reader is referred to Bertoin (1996) and Kyprianou (2006).

Assume the canonical decomposition of X is given by

$$X_t = -ct + \sigma B_t + J_t,\ t \geq 0, \tag{2.3}$$

where $\{B_t, t \geq 0\}$ is a standard Wiener process, $\{J_t, t \geq 0\}$ is a pure upward jump Lévy process that is independent of $\{B_t, t \geq 0\}$. In addition $J_0 = 0$. Note that the dual model with diffusion in Avanzi and Gerber (2008) corresponds to the case in which $\Pi(dx) = \lambda F(dx)$, where $\lambda > 0$ is the Poisson parameter and F is the distribution of

individual gains, and the rate of expenses is given by $c_0 = c + \int_0^1 x\Pi(dx)$. In particular, when $\sigma = 0$, the model deduces the so-called dual model in Avanzi, Gerber and Shiu (2007).

We now recall the definition of the $q-$scale function $W^{(q)}$. For each $q \geq 0$ there exits a continuous and increasing function $W^{(q)} : \mathbb{R} \to [0, \infty)$, called the q-scale function defined in such a way that $W^{(q)}(x) = 0$ for all $x < 0$ and on $[0, \infty)$ its Laplace transform is given by

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x) dx = \frac{1}{\psi(\theta) - q}, \ \theta > \Phi(q). \tag{2.4}$$

We will frequently use the following functions

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) dy, \ x \in \mathbb{R},$$

and

$$\overline{Z}^{(q)}(x) = \int_0^x Z^{(q)}(z) dz, \ x \in \mathbb{R}.$$

Note that

$$Z^{(q)}(x) = 1, \ \ \overline{Z}^{(q)}(x) = x, \quad x \leq 0.$$

The following facts about the scale functions are taken from Chan, Kyprianou and Savov (2011). If X has paths of bounded variation then, for all $q \geq 0$, $W^{(q)}|_{(0,\infty)} \in C^1(0, \infty)$ if and only if Π has no atoms. In the case that X has paths of unbounded variation, it is known that, for all $q \geq 0$, $W^{(q)}|_{(0,\infty)} \in C^1(0, \infty)$. Moreover if $\sigma > 0$ then $C^1(0, \infty)$ may be replaced by $C^2(0, \infty)$. Further, if the Lévy measure has a density, then the scale functions are always differentiable. It is well known that

$$W^{(\delta)}(0+) = \begin{cases} \frac{1}{c_0}, & \text{if } X \text{ has paths of bounded variation,} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$W^{(\delta)\prime}(0+) = \begin{cases} \frac{2}{\sigma^2}, & \text{if } \sigma \neq 0, \\ \frac{q + \Pi(0,\infty)}{c_0^2}, & \text{if } X \text{ is compound Poisson} \\ \infty, & \text{if } \sigma = 0 \text{ and } \Pi(0,\infty) = \infty. \end{cases}$$

In all cases, if $E(X_1) > 0$, then $W(\infty) = 1/E(X_1)$. If $q > 0$, then $W^{(q)}(x) \sim e^{\Phi(q)x}/\Psi'(\Phi(q))$ as $x \to \infty$.

We assume that the surplus process of the company is modeled by (2.3) if no dividends are paid. An admissible (dividend) strategy $\pi = \{L_t^\pi | t \geq 0\}$ is given by a nondecreasing, right-continuous and $\mathbb{F}$-adapted process starting at 0. Let $U^\pi = \{U_t^\pi : t \geq 0\}$ be the company's surplus, net of dividend payments, at time t. Thus,

$$U_t^\pi = X(t) - L_t^\pi, \quad t \geq 0.$$

In this article we are interested in the case that π only admits absolutely continuous strategies such that $dD_t^\pi = l^\pi(t)dt$, $l^\pi(t)$ satisfies $0 \leq l^\pi(t) \leq \alpha$, where α is a ceiling rate. Moreover, we make the assumption that, if X has paths of bounded variation, $\alpha < c + \int_0^1 y\Pi(dy)$. We define the dividend-value function V_π by

$$V_\pi(x) = E\left[\int_0^{\tau_\pi} e^{-qt} l^\pi(s)ds | U_0^\pi = x\right],$$

where $q > 0$ is an interest force for the calculation of the present value and τ_π is the time of ruin which is defined by

$$\tau_\pi = \inf\{t > 0 | U_t^\pi = 0\}.$$

We denote by Ξ the set of all the admissible dividend strategies. The objective is to solve the following stochastic control problem: the maximal dividend-value function, which is defined as

$$V(x) = \sup_{\pi \in \Xi} V_\pi(x), \tag{2.5}$$

and to find an optimal policy $\pi^* \in \Xi$ that satisfies $V(x) = V_{\pi^*}(x)$ for all $x \geq 0$. In this paper, we will prove that the optimal dividend strategy is formed by a threshold strategy with parameters b^* (the definition of b^* is given by (4.4)) and α: whenever the controlled risk process is below b^*, no dividends are paid; however, when the controlled risk process is above this level, dividends are paid continuously at the maximal admissible rate α.

3 Threshold dividend strategies

In this section, we assume that the company pays dividends according to the following threshold strategy governed by parameters $b > 0$ and $\alpha > 0$. Whenever the modified

surplus is below the threshold level b, no dividends are paid. However, when the surplus is above this threshold level, dividends are paid continuously at a constant rate α that does not exceed the rate of expense c. We define the modified risk process $U_b = \{U_b(t) : t \geq 0\}$ by $U_b(t) = X_t - D_b(t)$, where $D_b(t) = \alpha \int_0^t \mathbf{1}_{\{U_b(t)>b\}} dt$. Let D_b denote the present value of all dividends until time of ruin T,

$$D_b = \alpha \int_0^T e^{-qt} \mathbf{1}_{\{U_b(t)>b\}} dt$$

where $T = \inf\{t > 0 : U_b(t) = 0\}$ with $T = \infty$ if $U_b(t) > 0$ for all $t \geq 0$. Here $q > 0$ is the discount factor. Denote by $V(x, b)$ the expected discounted value of dividend payments, that is,

$$V(x, b) = E(D_b | U_b(0) = x).$$

Clearly, $0 \leq V(x, b) \leq \frac{\alpha}{q}$ and $\lim_{x\to\infty} V(x, b) = \frac{\alpha}{q}$.

Denote by Γ the extended generator of the process X, which acts on C^2 function g defined by

$$\mathcal{A}g(x) = \frac{1}{2}\sigma^2 g''(x) - cg'(x) + \int_0^\infty [g(x + y) - g(x) - g'(x)y\mathbf{1}_{\{|y|<1\}}]\Pi(dy). \tag{3.1}$$

Theorem 3.1. *Assume that, as a function of x, $V(x, b)$ is bounded and twice continuously differentiable on $(0, b) \cup (b, \infty)$ with a bounded first derivative. Then $V(x, b)$ satisfies the following integro-differential equations*

$$\mathcal{A}V(x, b) = qV(x, b), \ 0 < x < b, \tag{3.2}$$

$$\mathcal{A}V(x, b) - \alpha V'(x, b) = qV(x, b) - \alpha, \ x > b, \tag{3.3}$$

and initial condition $V(0, b) = 0$.

Proof Applying Itô's formula for semimartingales one has

$$E_x \left[e^{-q(t\wedge T)} V(U_b(t \wedge T), b)\right] = V(x, b)$$
$$+E_x \int_0^{t\wedge T} e^{-qs}[(\mathcal{A} - q)V(U_b(s), b) - \alpha \mathbf{1}_{\{U_b(s)>b\}} V'(U_b(s), b)]ds.$$

Letting $t \to \infty$ and note that $V(0, b) = 0$ we have

$$V(x, b) = \alpha E_x \int_0^T e^{-qt} \mathbf{1}_{\{U_b(t)>b\}} dt$$

if and only if

$$(\mathcal{A}-q)V(x,b)-\alpha\mathbf{1}_{\{x>b\}}V'(x,b)=-\alpha\mathbf{1}_{\{x>b\}}.$$

This ends the proof.

Remark 3.1. *From (3.2) and (3.3) one can prove that V satisfying the following*

$$V(b-,b)=V(b+,b)=V(b,b), \tag{3.4}$$

$$(c_1+\alpha)V'(b+,b)-c_1V'(b-,b)=\alpha,\ \text{ if } \sigma=0, \tag{3.5}$$

$$V'(b+,b)=V'(b-,b),\ \text{ if } \sigma\neq 0. \tag{3.6}$$

Define the first passage times, with the convention $\inf\emptyset=\infty$,

$$T_b^+=\inf\{t\geq 0: U_b(t)>b\},\quad T_b^-=\inf\{t\geq 0: U_b(t)\leq b\}.$$

For $q\geq 0$, let

$$\Phi_1(q)=\sup\{\theta\geq 0|\Psi(\theta)+\alpha\theta=q\}.$$

The following result generalized the result of Ng (2009, Theorem 2) in which only the dual of the classical insurance risk model was considered.

Theorem 3.2. *For $x>b$,*

$$V(x,b)=\frac{\alpha}{q}+\left(V(b,b)-\frac{\alpha}{q}\right)e^{-\Phi_1(q)(x-b)}. \tag{3.7}$$

Proof By using the strong Markov property of U_b at T_b^- as in Yin et al. (2013), we have

$$\begin{aligned} V(x,b) &= \frac{\alpha}{q}-\frac{\alpha}{q}E_x(e^{-qT_b^-},T_b^-<\infty) \\ &\quad +E_x(e^{-qT_b^-}V(U_b(T_b^-),b),T_b^-<\infty). \end{aligned}$$

The result (3.4) follows since $P_x(U_b(T_b^-)=b)=1$ and

$$E_x(e^{-qT_b^-},T_b^-<\infty)=\exp(-\Phi_1(q)(x-b)).$$

Theorem 3.3. *For* $0 < x < b$,

$$\begin{aligned} V(x,b) &= -\frac{1}{2}\sigma^2 W^{(q)}(b-x) \\ &+ V(b,b)\Bigg(e^{\Phi_1(q)(b-x)} - \frac{1}{2}\sigma^2\Phi_1(q)W^{(q)}(b-x) \\ &\quad +\alpha\Phi_1(q)e^{\Phi_1(q)(b-x)}\int_0^{b-x} W^{(q)}(z)e^{-\Phi_1(q)z}dz\Bigg) \\ &+ \frac{\alpha}{q}\left(\frac{1}{2}\sigma^2\Phi_1(q)W^{(q)}(b-x) - e^{\Phi_1(q)(b-x)}\Phi_1(q)\int_0^{b-x} Z^{(q)}(z)e^{-\Phi_1(q)z}dz\right) \\ &+ \frac{\alpha}{q}e^{\Phi_1(q)(b-x)}\left(q-\alpha\Phi_1(q)\right)\int_0^{b-x} W^{(q)}(z)e^{-\Phi_1(q)z}dz. \end{aligned} \tag{3.8}$$

where

$$V(b,b) = \frac{\frac{1}{2}\sigma^2 W^{(q)}(b) - B(b) - C(b)}{A(b)}.$$

Here

$$A(b) = e^{\Phi_1(q)(b)} - \frac{1}{2}\sigma^2\Phi_1(q)W^{(q)}(b) + \alpha\Phi_1(q)e^{\Phi_1(q)(b)}\int_0^b W^{(q)}(z)e^{-\Phi_1(q)z}dz,$$

$$B(b) = \frac{\alpha}{q}\left(\frac{1}{2}\sigma^2\Phi_1(q)W^{(q)}(b) - e^{\Phi_1(q)(b)}\Phi_1(q)\int_0^b Z^{(q)}(z)e^{-\Phi_1(q)z}dz\right),$$

$$C(b) = \frac{\alpha}{q}e^{\Phi_1(q)(b)}\left(q-\alpha\Phi_1(q)\right)\int_0^b W^{(q)}(z)e^{-\Phi_1(q)z}dz.$$

In particular, when $\sigma = 0$,

$$\begin{aligned} V(x,b) &= V(b,b)e^{\Phi_1(q)(b-x)}\left(1+\alpha\Phi_1(q)\int_0^{b-x} W^{(q)}(z)e^{-\Phi_1(q)z}dz\right) \\ &+ \frac{\alpha}{q}e^{\Phi_1(q)(b-x)}\left(q-\alpha\Phi_1(q)\right)\int_0^{b-x} W^{(q)}(z)e^{-\Phi_1(q)z}dz \\ &- \frac{\alpha}{q}e^{\Phi_1(q)(b-x)}\Phi_1(q)\int_0^{b-x} Z^{(q)}(z)e^{-\Phi_1(q)z}dz, \end{aligned} \tag{3.9}$$

where

$$V(b,b) = \frac{\frac{\alpha}{q}\Phi_1(q)\int_0^b Z^{(q)}(z)e^{-\Phi_1(q)z}dz - \frac{\alpha}{q}\left(q-\alpha\Phi_1(q)\right)\int_0^b W^{(q)}(z)e^{-\Phi_1(q)z}dz}{1+\alpha\Phi_1(q)\int_0^b W^{(q)}(z)e^{-\Phi_1(q)z}dz}.$$

Proof We first assume that $\lambda := \int_0^\infty \Pi(dy) < \infty$. Set $c_0 = c + \int_0^1 y\Pi(dy)$ and $f(y)dy = \frac{\Pi(dy)}{\lambda}$, then f is a probability density on $(0,\infty)$. In this case, in view of (3.2),

the integro-differential equation (3.7) can be written as

$$\begin{aligned}\frac{1}{2}\sigma^2 V''(x,b) - c_0 V'(x,b) &= -\lambda\left[V(b,b)-\frac{\alpha}{q}\right]\int_{b-x}^{\infty}\exp(-\Phi_1(q)(x+y-b))f(y)dy \\ &\quad -\lambda\int_0^{b-x} V(x+y,b)f(y)dy - \frac{\lambda\alpha}{q}(1-F(b-x)) \\ &\quad +(\lambda+q)V(x,b),\ 0<x<b, \qquad (3.10)\end{aligned}$$

where F is the distribution function of f. Replace the variable x by $z = b - x$, and define W by $W(z,b) = V(b-z,b), 0<z<b$. The (3.10) becomes

$$\begin{aligned}\frac{1}{2}\sigma^2 W''(z,b) + c_0 W'(z,b) &= -\lambda\left[W(0,b)-\frac{\alpha}{q}\right]\int_{z}^{\infty}\exp(-\Phi_1(q)(y-z))f(y)dy \\ &\quad -\lambda\int_0^{z} W(y,b))f(z-y)dy - \frac{\lambda\alpha}{q}(1-F(z)) \\ &\quad +(\lambda+q)W(z,b),\ 0<z<b, \qquad (3.11)\end{aligned}$$

with initial condition $W(0,b) = V(b,b)$ and boundary condition $W(b,b) = 0$. We extend the definition of W by (3.11) to $z \geq 0$ and denote the resulting function by w. Then we have

$$\begin{aligned}\frac{1}{2}\sigma^2 w''(z) + c_0 w'(z) &= -\lambda\left[w(0)-\frac{\alpha}{q}\right]\int_{z}^{\infty}\exp(-\Phi_1(q)(y-z))f(y)dy \\ &\quad -\lambda\int_0^{z} w(y))f(z-y)dy - \frac{\lambda\alpha}{q}(1-F(z)) \\ &\quad +(\lambda+q)w(z),\ z\geq 0, \qquad (3.12)\end{aligned}$$

with $w(0) = V(b,b)$ and $w(b) = 0$.

For a function g, denoted by $\hat{g}$ the Laplace transform of g, i.e. $\hat{g}(\xi) = \int_0^\infty e^{-\xi y} g(y)dy$. Then the Laplace transform $\hat{w}$ for w can be easily determined from Eq. (3.11) as

$$\hat{w}(\xi) = \frac{\frac{1}{2}\sigma^2(-1+\xi w(0)) + c_0 w(0) + \frac{\lambda\alpha}{q\xi}(\hat{f}(\xi)-1) - \frac{\lambda}{\xi-\Phi_1(q)}(w(0)-\frac{\alpha}{q})(\hat{f}(\Phi_1(q))-\hat{f}(\xi))}{\frac{1}{2}\sigma^2\xi^2 + c_0\xi - (\lambda+q) + \lambda\hat{f}(\xi)}. \qquad (3.13)$$

Note that

$$\int_0^\infty e^{-x\xi} W^{(q)}(x)dx = \frac{1}{\frac{1}{2}\sigma^2\xi^2 + c_0\xi - (\lambda+q) + \lambda\hat{f}(\xi)},$$

$$\int_0^\infty e^{-x\xi}dx \int_0^x W^{(q)}(y)dy = \frac{1}{\xi(\frac{1}{2}\sigma^2\xi^2 + c_0\xi - (\lambda+q) + \lambda\hat{f}(\xi))},$$

$$\int_0^\infty e^{-x\xi}dx\int_0^x dy\int_0^y W^{(q)}(z)dz = \frac{1}{\xi^2(\frac{1}{2}\sigma^2\xi^2 + c_0\xi - (\lambda+q) + \lambda\hat{f}(\xi))}.$$

Now inverting (3.13) gives

$$\begin{aligned}
w(z) &= -\frac{1}{2}\sigma^2 W^{(q)}(z) + \frac{1}{2}\sigma^2 w(0)\int_0^z W^{(q)}(z-y)\delta_0'(y)dy \\
&\quad + c_0 w(0)W^{(q)}(z) + \lambda w(0)\left(W^{(q)} * (f - \hat{f}(\Phi_1(q))\delta_0) * l\right)(z) \\
&\quad + \frac{\lambda\alpha}{q}\left(\Phi_1(q)\frac{Z^{(q)}-1}{q} * (\delta_0 - f) * l\right)(z) \\
&\quad + \frac{\lambda\alpha}{q}\hat{f}(\Phi_1(q)) - 1)(W^{(q)} * l)(z), \qquad (3.14)
\end{aligned}$$

where δ_0 is the Dirac delta function at 0, $h_1 * h_2$ stands for convolution of h_1 and h_2 and $l(z) = \exp(\Phi_1(q)z)$. After some tedious calculations, we get

$$\int_0^z W^{(q)}(z-y)\delta_0'(y)dy = {W^{(q)}}'(z),$$

$$\begin{aligned}
\lambda\left(W^{(q)} * (f - \hat{f}(\Phi_1(q))\delta_0) * l\right)(z) &= \alpha\Phi_1(q)\int_0^z W^{(q)}(z-y)e^{\Phi_1(q)y}dy \\
&\quad + e^{\Phi_1(q)z} - \frac{1}{2}\sigma^2 {W^{(q)}}'(z) \\
&\quad - (\frac{1}{2}\sigma^2\Phi_1(q) + c_0)W^{(q)}(z),
\end{aligned}$$

$$\begin{aligned}
\lambda(Z^{(q)} - 1) * (\delta_0 - f) * l)(z) &= \left(\frac{1}{2}\sigma^2 q\Phi_1(q) + c_0 q\right)\int_0^z W^{(q)}(z-y)e^{\Phi_1(q)y}dy \\
&\quad + \frac{1}{2}\sigma^2 q W^{(q)}(z) - q\int_0^z Z^{(q)}(z-y)e^{\Phi_1(q)y}dy.
\end{aligned}$$

Substituting the three expressions above into (3.14) we arrive at

$$\begin{aligned}
w(z) &= -\frac{1}{2}\sigma^2 W^{(q)}(z) \\
&\quad + w(0)\left(e^{\Phi_1(q)z} - \frac{1}{2}\sigma^2\Phi_1(q)W^{(q)}(z)\right. \\
&\quad \left. + \alpha\Phi_1(q)e^{\Phi_1(q)z}\int_0^z W^{(q)}(y)e^{-\Phi_1(q)y}dy\right) \\
&\quad + \frac{\alpha}{q}\left(\frac{1}{2}\sigma^2\Phi_1(q)W^{(q)}z - e^{\Phi_1(q)z}\Phi_1(q)\int_0^z Z^{(q)}(y)e^{-\Phi_1(q)y}dy\right) \\
&\quad + \frac{\alpha}{q}e^{\Phi_1(q)z}(q - \alpha\Phi_1(q))\int_0^z W^{(q)}(y)e^{-\Phi_1(q)y}dy, \qquad (3.15)
\end{aligned}$$

and the result (3.8) follows since $V(x,b) = w(b-x)$ and $w(0) = V(b,b)$.

Now, we assume that $\lambda := \int_0^\infty \Pi(dy) = \infty$. Let Π_n be measures on $(0,\infty)$ defined by

$$\Pi_n(dx) = \Pi(dx)\mathbf{1}_{\{(\frac{1}{n},\infty)\}}(x),\ n \geq 1.$$

Then we have

$$\lambda_n := \int_0^\infty \Pi_n(dx) \leq n^2 \int_{\frac{1}{n}}^1 x^2\Pi(dx) + \int_1^\infty (1 \wedge x^2)\Pi(dx) < \infty.$$

Set $c_n = c + \int_0^1 y\Pi_n(dy)$ and $f_n(y)dy = \frac{\Pi_n(dy)}{\lambda_n}$, then for each $n \geq 1$, f_n is a probability density on $(0,\infty)$. Similar to (3.10) we consider the following integro-differential equation

$$\begin{aligned}\frac{1}{2}\sigma^2 V_n''(x,b) - c_n V_n'(x,b) &= -\lambda\left[V_n(b,b) - \frac{\alpha}{q}\right]\int_{b-x}^\infty \exp(-\Theta_n(q)(x+y-b))f_n(y)dy \\ &\quad -\lambda\int_0^{b-x} V_n(x+y,b)f_n(y)dy - \frac{\lambda\alpha}{q}(1-F_n(b-x)) \\ &\quad +(\lambda+q)V_n(x,b),\ 0<x<b, \end{aligned} \tag{3.16}$$

where $\Theta_n(q) = \sup\{\theta \geq 0 | \Psi_n(\theta) + \alpha\theta = q\}$. Here,

$$\Psi_n(\theta) = c_n\theta + \frac{1}{2}\sigma^2\theta^2 + \int_0^\infty (e^{-\theta x} - 1 + \theta x\mathbf{1}_{\{|x|<1\}})\Pi_n(dx). \tag{3.17}$$

Repeating the same argument as the case that $\lambda := \int_0^\infty \Pi(dy) < \infty$, we obtain

$$\begin{aligned} V_n(x,b) &= -\frac{1}{2}\sigma^2 W_n^{(q)}(b-x) \\ &+ V_n(b,b)\left(e^{\Theta_n(q)(b-x)} - \frac{1}{2}\sigma^2\Theta_n(q)W_n^{(q)}(b-x)\right. \\ &\quad \left. +\alpha\Theta_n(q)e^{\Theta_n(q)(b-x)}\int_0^{b-x} W_n^{(q)}(z)e^{-\Theta_n(q)z}dz\right) \\ &+ \frac{\alpha}{q}\left(\frac{1}{2}\sigma^2\Theta_n(q)W_n^{(q)}(b-x) - e^{\Theta_n(q)(b-x)}\Theta_n(q)\int_0^{b-x} Z_n^{(q)}(z)e^{-\Theta_n(q)z}dz\right) \\ &+ \frac{\alpha}{q}e^{\Theta_n(q)(b-x)}(q - \alpha\Theta_n(q))\int_0^{b-x} W_n^{(q)}(z)e^{-\Theta_n(q)z}dz, \end{aligned} \tag{3.18}$$

where $W_n^{(q)}$ and $Z_n^{(q)}$ are scale functions corresponding to the process X_n with Laplace exponent Ψ_n. Since $\lim_{n\to\infty}\Psi_n(\theta) = \Psi(\theta)$, then $X_n \to X$ weakly as $n \to \infty$. Thus $\lim_{n\to\infty} V_n = V$, $\lim_{n\to\infty} W_n^{(q)} = W^{(q)}$, $\lim_{n\to\infty} Z_n^{(q)} = Z^{(q)}$ and $\lim_{n\to\infty}\Theta_n = \Phi_1$. Consequently, (3.8) still holds for this case. This ends the proof of Theorem 3.3.

Remark 3.2. *For $\sigma \geq 0$, from the graph of*

$$\Psi(\Phi_1(q)) + \alpha\Phi_1(q) = q$$

one can verify that $\Phi_1(q) \to 0$ *when* $\alpha \to \infty$. *After some simple calculations we get* $\lim_{\alpha\to\infty} \alpha\Phi_1(q) = q$,

$$\lim_{\alpha\to\infty} \alpha(q - \alpha\Phi_1(q)) = q\Psi'(0+),$$

and

$$\lim_{\alpha\to\infty} V(b,b) = \frac{\overline{Z}^{(q)}(b)}{Z^{(q)}(b)} + \frac{\Psi'(0+)}{qZ^{(q)}(b)} - \frac{\Psi'(0+)}{q}.$$

As a result, for $0 < x < b$, *we arrive at*

$$\lim_{\alpha\to\infty} V(x,b) = \frac{\overline{Z}^{(q)}(b)}{Z^{(q)}(b)} Z^{(q)}(b-x) - \overline{Z}^{(q)}(b-x) + \frac{\Psi'(0+)}{q}\left(\frac{Z^{(q)}(b-x)}{Z^{(q)}(b)} - 1\right),$$

which is the expected discounted value of dividend payments for the barrier strategy. See Lemma 2.1 of Bayraktar, Kyprianou and Yamazaki (2013).

4 Optimal dividend strategy

Suppose the maximal dividend-value function V is $C^2(0,\infty)$ (resp. $C^1(0,\infty)$ when X is of unbounded (resp. bounded) variation. Standard Markovian arguments, see Fleming and Soner(1993), formally yield that $V(x)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\Gamma V(x) - qV(x) + \sup_{0\le r\le\alpha} \{r(1 - V'(x))\} = 0 \tag{4.1}$$

with $V(0) = 0$, where Γ is the extended generator of the process X, which defined by

$$\Gamma g(x) = \frac{1}{2}\sigma^2 g''(x) - cg'(x) + \int_0^\infty [g(x+y) - g(x) - g'(x)y\mathbf{1}_{\{|y|<1\}}]\Pi(dy). \tag{4.2}$$

It follows from the HJB equation (4.1) that an optimal dividend policy has to fulfil $r = 0$ if $V'(U^\pi_{t-}) > 1$; $r = \alpha$ if $V'(U^\pi_{t-}) < 1$. In some situations the optimal dividend strategy is a threshold strategy. In fact if $V'(x,0) < 1$ for $x > 0$, then the threshold strategy with $b^* = 0$ is optimal, if $V'(x,b^*) > 1$ for $x < b^*$ and $V'(x,b^*) < 1$ for $x > b^*$, then the threshold strategy with $b^* > 0$ is optimal. The optimal threshold b^* can be obtained by $V'(b^*,b^*) = 1$. From those facts one sees that if $V(x,b^*)$ is a concave function on $(0,\infty)$, then the optimal dividend strategy is a threshold strategy.

It follows from (3.7) that

$$V'(x,b) = -\Phi_1(q)\left(V(b,b) - \frac{\alpha}{q}\right)e^{-\Phi_1(q)(x-b)},\ x > b. \tag{4.3}$$

If $\Phi_1(q)\frac{\alpha}{q} \le 1$, then $V'(x,0) \le 1$ since $V(0,0) = 0$. Thus $b^* = 0$. Now, suppose that $\Phi_1(q)\frac{\alpha}{q} > 1$, we get the condition for b^*:

$$-\Phi_1(q)\left(V(b^*,b^*) - \frac{\alpha}{q}\right) = 1.$$

Or, equivalently, b^* is the solution of the equation

$$V(b^*,b^*) = \frac{\alpha}{q} - \frac{1}{\Phi_1(q)}. \tag{4.4}$$

From (3.7) and (3.8) we get

$$V(x,b^*) = \begin{cases} \frac{\alpha}{q} - \frac{1}{\Phi_1(q)}e^{-\Phi_1(q)(x-b^*)}, & \text{if } x > b^*, \\ -\frac{\alpha\int_0^{b^*-x} W^{(q)}(z)e^{-\Phi_1(q)z}dz}{e^{-\Phi_1(q)(b^*-x)}} + \frac{\alpha}{q}Z^{(q)}(b^*-x) - \frac{e^{\Phi_1(q)(b^*-x)}}{\Phi_1(q)}, & \text{if } 0 < x < b^*. \end{cases} \tag{4.5}$$

Taking derivative with respect to x in the both sides of the above relation leads to

$$V'(x,b^*) = \begin{cases} e^{-\Phi_1(q)(x-b^*)}, & \text{if } x > b^*, \\ e^{\Phi_1(q)(b^*-x)}\left(1 + \alpha\Phi_1(q)\int_0^{b^*-x} W^{(q)}(z)e^{-\Phi_1(q)z}dz\right), & \text{if } 0 < x < b^*. \end{cases} \tag{4.6}$$

It follows that $V'(x,b^*) < 1$ when $x > b^*$. Further, for $0 < x < b^*$,

$$\begin{aligned} V''(x,b^*) &= -\Phi_1(q)e^{\Phi_1(q)(b^*-x)}\left(1 + \alpha\Phi_1(q)\int_0^{b^*-x} W^{(q)}(z)e^{-\Phi_1(q)z}dz\right) \\ &- \alpha\Phi_1(q)W^{(q)}(b^*-x) < 0. \end{aligned} \tag{4.7}$$

Thus for $0 < x < b^*$, $V'(x,b^*) > V'(b^*,b^*) = 1$.

Remark 4.1. *From (4.6) and (4.7) we find that $V'(x,b^*)$ is continuous on $(0,\infty)$, $V''(b^*-,b^*) = -\Phi_1(q) - \alpha\Phi_1(q)W^{(q)}(0)$ and $V''(b^*+,b^*) = -\Phi_1(q)$. So that if $\sigma = 0$, then $V''(b^*-,b^*) \neq V''(b^*+,b^*)$ and, if $\sigma \neq 0$, then $V''(b^*-,b^*) = V''(b^*+,b^*)$.*

In summary, we have the following:

Theorem 4.1. *Consider the stochastic control problem (2.5). We have $V(x) = V(x,b^*)$ as defined in (4.5) and the threshold strategy with threshold b^* is the optimal dividend strategy over Ξ.*

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On the optimal dividend problem for a spectrally positive Lévy process

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Abstract

In this paper we study the optimal dividend problem for a company whose surplus process evolves as a spectrally positive Lévy process before dividends are deducted. This model including the dual model of the classical risk model and the dual model with diffusion as special cases. We assume that dividends are paid to the shareholders according to admissible strategy whose dividend rate is bounded by a constant. The objective is to find a dividend policy so as to maximize the expected discounted value of dividends which are paid to the shareholders until the company is ruined. We show that the optimal dividend strategy is formed by a threshold strategy.

Key Words: Threshold strategy, Dual model, Optimal dividend strategy, Scale functions, Spectrally positive Lévy process, Stochastic control.

1 INTRODUCTION

Recently, dividend optimization problems for financial and insurance corporations have attracted extensive attention. How should corporation pay dividends to its shareholders? A possible goal is that the company tries to maximize the expectation of the discounted dividends until possible ruin of the company. In recent years, quite a few interesting papers deal with the optimal dividend problem in the dual model. For example, Avanzi et al. (2007) consider the model which is dual to the classical risk mode, the authors studied how the expectation of the discounted dividends until ruin can be calculated when gain distribution has a exponential distribution or mixtures of exponential distributions and shows how the exact value of the optimal dividend barrier can be determined, if the jump distribution is known. Avanzi and Gerber (2008) examined the same problem for the dual model perturbed by diffusion. Moreover, they pointed out that "the optimal dividend strategy in the dual model is a barrier strategy. A direct proof that optimal strategy is a barrier strategy is of some interest but has not been given to our knowledge; the proof in Bayraktar and Egami (2008) is for exponential gains only." Yao et al. (2010) considered the optimal problem with dividend payments and issuance of equity in a dual risk model without a diffusion, assuming proportional transaction costs, they found optimal strategy which maximizes the expected present value of the dividends payout minus the discounted costs of issuing new equity before ruin. In addition, for exponentially distributed jump sizes, closed form solutions are obtained. Dai, et al. (2010, 2011) considered the same problem as in Yao et al. (2010) for a dual risk model with a diffusion with bounded gains and exponential gains, respectively. Avanzi, Shen and Wong (2011) determined an explicit form for the value function in the dual model with diffusion when the gains distribution is a mixture of exponentials. They showed that a barrier dividend strategy is also optimal and conjectured that the optimal dividend strategy in the dual model with diffusion should be the barrier strategy, regardless of the gains distribution. Until recently, Bayraktar, Kyprianou and Yamazaki (2013) using the fluctuation theory of spectrally positive Lévy processes, show the optimality of barrier strategies for all such Lévy processes.

Barrier strategies often serve as candidates for the optimal strategy when the dividend rate is unrestricted. However, if a barrier strategy is applied, ultimate ruin of the company is certain. In many circumstances this is not desirable. This consideration leads us to impose restriction on the dividend stream. Ng (2009) considered the dual of the compound Poisson model under a threshold dividend strategy and derived a set of two integro-differential equations satisfied by the expected total discounted dividends until ruin and showed how the equations can be solved by using only one of the two integro-differential equations. The cases where profits follow an exponential or a mixture of exponential distributions are then solved and the discussion for the case of a general profit distribution follows by the use of Laplace transforms. He illustrated how the optimal threshold level that maximizes the expected total discounted dividends until ruin can be obtained. In this paper we provide a uniform mathematical framework to analyze the optimal control problem with dividends payout for a general spectrally positive Lévy process. We assume that dividends are paid to the shareholders according to admissible strategies whose dividend rate is bounded by a constant. Under this additional constraint, we show that the optimal dividend strategy is formed by a threshold strategy. This problem has been considered by Asmussen and Taksar (1997), Jeanblanc-Picqué and Shiryaev (1995) and Hϕjgaard, B. and Taksar, M. (1999) in the diffusive case.

The rest of the paper is organized as follows. Section 2 presents the model and formulates the dividend optimization problem. Section 3 discusses the threshold strategies. Explicit expressions for the expected discounted value of dividend payments are obtained, and in Section 4 we give the main results, it is shown that the optimal dividend strategy is formed by a threshold strategy. This strategy also called the refraction strategy, prescribes paying no dividends when the net surplus of the company is below an optimal barrier b^*, and paying dividends at the fixed maximal rate α when net surplus exceeds b^*.

2 The model and the optimization problem

Let $X = \{X_t\}_{t\geq 0}$ be a spectrally positive Lévy process with non-monotone paths on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is generated by the process X and satisfies the usual conditions. The Lévy triplet of X is given by (c, σ, Π), where $c > 0$, $\sigma \geq 0$ and Π is a measure on $(0, \infty)$ satisfying

$$\int_0^\infty (1 \wedge x^2)\Pi(dx) < \infty.$$

Denote by P_x for the law of X when $X_0 = x$. Let E_x be the expectation associated with P_x. For short, we write P and E when $X_0 = 0$. The Laplace exponent of X is given by

$$\Psi(\theta) = \frac{1}{t}\log Ee^{-\theta X_t} = c\theta + \frac{1}{2}\sigma^2\theta^2 + \int_0^\infty (e^{-\theta x} - 1 + \theta x \mathbf{1}_{\{|x|<1\}})\Pi(dx), \tag{2.1}$$

where $\mathbf{1}_A$ is the indicator function of a set A. In the sequel, we assume that $-\Psi'(0+) = \mathbb{E}(X_1) > 0$ which implies the process X drifts to $+\infty$. It is well known that if $\int_1^\infty y\Pi(dy) < \infty$, then $\mathbb{E}(X_1) < \infty$, and $\mathbb{E}(X_1) = -c + \int_1^\infty y\Pi(dy)$. Note that X has paths of bounded variation if and only if

$$\sigma = 0 \ \text{ and } \ \int_0^\infty (1 \wedge x)\Pi(dx) < \infty.$$

In this case, the Laplace exponent (2.1) can be written as

$$\Psi(\theta) = c_0\theta + \frac{1}{2}\sigma^2\theta^2 + \int_0^\infty (e^{-\theta x} - 1)\Pi(dx), \tag{2.2}$$

with $c_0 = c + \int_0^1 x\Pi(dx)$ the so-called drift of X.

For an arbitrary spectrally positive Lévy process, the Laplace exponent Ψ is strictly convex and $\lim_{\theta\to\infty}\Psi(\theta) = \infty$. Thus there exists a function $\Phi : [0, \infty) \to [0, \infty)$ defined by $\Phi(q) = \sup\{\theta \geq 0 | \Psi(\theta) = q\}$ such that $\Psi(\Phi(q)) = q,\ q \geq 0$.

For more details on spectrally positive Lévy processes, the reader is referred to Bertoin (1996) and Kyprianou (2006).

Assume the canonical decomposition of X is given by

$$X_t = -ct + \sigma B_t + J_t,\ t \geq 0, \tag{2.3}$$

where $\{B_t, t \geq 0\}$ is a standard Wiener process, $\{J_t, t \geq 0\}$ is a pure upward jump Lévy process that is independent of $\{B_t, t \geq 0\}$. In addition $J_0 = 0$. Note that the dual model with diffusion in Avanzi and Gerber (2008) corresponds to the case in which $\Pi(dx) = \lambda F(dx)$, where $\lambda > 0$ is the Poisson parameter and F is the distribution of individual gains, and the rate of expenses is given by $c_0 = c + \int_0^1 x\Pi(dx)$. In particular, when $\sigma = 0$, the model reduces to the so-called dual model in Avanzi, Gerber and Shiu (2007).

We now recall the definition of the $q-$scale function $W^{(q)}$. For each $q \geq 0$ there exits a continuous and increasing function $W^{(q)} : \mathbb{R} \to [0, \infty)$, called the q-scale function defined in such a way that $W^{(q)}(x) = 0$ for all $x < 0$ and on $[0, \infty)$ its Laplace transform is given by

$$\int_0^\infty \mathrm{e}^{-\theta x} W^{(q)}(x)dx = \frac{1}{\psi(\theta) - q}, \;\; \theta > \Phi(q). \tag{2.4}$$

Closely related to $W^{(q)}$ is the function $Z^{(q)}$ given by

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y)dy, \;\; x \in \mathbb{R}.$$

We will frequently use the following function

$$\overline{Z}^{(q)}(x) = \int_0^x Z^{(q)}(z)dz, \;\; x \in \mathbb{R}.$$

Note that

$$Z^{(q)}(x) = 1, \;\; \overline{Z}^{(q)}(x) = x, \quad x \leq 0.$$

The following facts about the scale functions are taken from Chan, Kyprianou and Savov (2011). If X has paths of bounded variation then, for all $q \geq 0$, $W^{(q)}|_{(0,\infty)} \in C^1(0, \infty)$ if and only if Π has no atoms. In the case that X has paths of unbounded variation, it is known that, for all $q \geq 0$, $W^{(q)}|_{(0,\infty)} \in C^1(0, \infty)$. Moreover if $\sigma > 0$ then $C^1(0, \infty)$ may be replaced by $C^2(0, \infty)$. Further, if the Lévy measure has a density, then the scale functions are always differentiable. It is well known that

$$W^{(\delta)}(0+) = \begin{cases} \frac{1}{c_0}, & \text{if } X \text{ has paths of bounded variation,} \\ 0, & \text{otherwise,} \end{cases}$$

and

$$W^{(\delta)\prime}(0+) = \begin{cases} \frac{2}{\sigma^2}, & \text{if } \sigma \neq 0, \\ \frac{q+\Pi(0,\infty)}{c_0^2}, & \text{if } X \text{ is compound Poisson} \\ \infty, & \text{if } \sigma = 0 \text{ and } \Pi(0,\infty) = \infty. \end{cases}$$

In all cases, if $E(X_1) > 0$, then $W(\infty) = 1/E(X_1)$. If $q > 0$, then $W^{(q)}(x) \sim e^{\Phi(q)x}/\Psi'(\Phi(q))$ as $x \to \infty$.

We assume that the surplus process of the company is modeled by (2.3) if no dividends are paid. An admissible (dividend) strategy $\pi = \{L_t^\pi | t \geq 0\}$ is given by a nondecreasing, right-continuous and $\mathbb{F}$-adapted process starting at 0. Let $U^\pi = \{U_t^\pi : t \geq 0\}$ be the company's surplus, net of dividend payments, at time t. Thus,

$$U_t^\pi = X(t) - L_t^\pi, \quad t \geq 0.$$

In this article we are interested in the case that π only admits absolutely continuous strategies such that

$$dD_t^\pi = l^\pi(t)dt, \tag{2.5}$$

and for $t \geq 0$, $l^\pi(t)$ satisfies

$$0 \leq l^\pi(t) \leq \alpha, \tag{2.6}$$

where α is a ceiling rate. Moreover, we make the assumption that, if X has paths of bounded variation,

$$\alpha < c + \int_0^1 y\Pi(dy). \tag{2.7}$$

We define the dividend-value function V_π by

$$V_\pi(x) = E\left[\int_0^{\tau_\pi} e^{-qt} l^\pi(s)ds | U_0^\pi = x\right],$$

where $q > 0$ is an interest force for the calculation of the present value and τ_π is the time of ruin which is defined by

$$\tau_\pi = \inf\{t > 0 | U_t^\pi = 0\}.$$

We denote by Ξ the set of all the admissible dividend strategies. The objective is to solve the following stochastic control problem: the maximal dividend-value function, which is defined as

$$V(x) = \sup_{\pi \in \Xi} V_\pi(x), \tag{2.8}$$

and to find an optimal policy $\pi^* \in \Xi$ that satisfies $V(x) = V_{\pi^*}(x)$ for all $x \geq 0$. In this paper, we will prove that the optimal dividend strategy is formed by a threshold strategy with parameters b^* (the definition of b^* is given by (4.4)) and α: whenever the controlled risk process is below b^*, no dividends are paid; however, when the controlled risk process is above this level, dividends are paid continuously at the maximal admissible rate α.

3 Threshold dividend strategies

In this section, we assume that the company pays dividends according to the following threshold strategy governed by parameters $b > 0$ and $\alpha > 0$. Whenever the modified surplus is below the threshold level b, no dividends are paid. However, when the surplus is above this threshold level, dividends are paid continuously at a constant rate α that does not exceed the rate of expense c. We define the modified risk process $U_b = \{U_b(t) : t \geq 0\}$ by $U_b(t) = X_t - D_b(t)$, where $D_b(t) = \alpha \int_0^t \mathbf{1}_{\{U_b(t)>b\}} dt$. Let D_b denote the present value of all dividends until time of ruin T,

$$D_b = \alpha \int_0^T e^{-qt} \mathbf{1}_{\{U_b(t)>b\}} dt$$

where $T = \inf\{t > 0 : U_b(t) = 0\}$ with $T = \infty$ if $U_b(t) > 0$ for all $t \geq 0$. Here $q > 0$ is the discount factor. Denote by $V(x, b)$ the expected discounted value of dividend payments, that is,

$$V(x, b) = E(D_b | U_b(0) = x).$$

Clearly, $0 \leq V(x, b) \leq \frac{\alpha}{q}$ and $\lim_{x\to\infty} V(x, b) = \frac{\alpha}{q}$.

Denote by Γ the extended generator of the process X, which acts on C^2 function g defined by

$$\mathcal{A}g(x) = \frac{1}{2}\sigma^2 g''(x) - cg'(x) + \int_0^\infty [g(x + y) - g(x) - g'(x)y\mathbf{1}_{\{|y|<1\}}]\Pi(dy). \tag{3.1}$$

Theorem 3.1. *Assume that, as a function of x, $V(x, b)$ is bounded and twice continuously differentiable on $(0, b) \cup (b, \infty)$ with a bounded first derivative. Then $V(x, b)$ satisfies the*

following integro-differential equations

$$\mathcal{A}V(x,b) = qV(x,b),\ 0 < x < b, \tag{3.2}$$

$$\mathcal{A}V(x,b) - \alpha V'(x,b) = qV(x,b) - \alpha,\ x > b, \tag{3.3}$$

and initial condition $V(0,b) = 0$.

Proof Applying Itô's formula for semimartingales one has

$$E_x\left[e^{-q(t\wedge T)}V(U_b(t\wedge T),b)\right] = V(x,b)$$
$$+E_x\int_0^{t\wedge T} e^{-qs}[(\mathcal{A}-q)V(U_b(s),b) - \alpha\mathbf{1}_{\{U_b(s)>b\}}V'(U_b(s),b)]ds.$$

Letting $t \to \infty$ and note that $V(0,b) = 0$ we have

$$V(x,b) = \alpha E_x\int_0^T e^{-qt}\mathbf{1}_{\{U_b(t)>b\}}dt$$

if and only if

$$(\mathcal{A}-q)V(x,b) - \alpha\mathbf{1}_{\{x>b\}}V'(x,b) = -\alpha\mathbf{1}_{\{x>b\}}.$$

This ends the proof.

Remark 3.1. *From (3.2) and (3.3) one can prove that* V *satisfying the following*

$$V(b-,b) = V(b+,b) = V(b,b), \tag{3.4}$$
$$(c_1+\alpha)V'(b+,b) - c_1V'(b-,b) = \alpha,\ \text{if } \sigma = 0, \tag{3.5}$$
$$V'(b+,b) = V'(b-,b),\ \text{if } \sigma \neq 0. \tag{3.6}$$

Define the first passage times, with the convention $\inf\emptyset = \infty$,

$$T_b^+ = \inf\{t \geq 0 : U_b(t) > b\},\quad T_b^- = \inf\{t \geq 0 : U_b(t) < b\}.$$

For $q \geq 0$, let

$$\Phi_1(q) = \sup\{\theta \geq 0 | \Psi(\theta) + \alpha\theta = q\}.$$

The following result generalized the result of Ng (2009, Theorem 2) in which only the dual of the classical insurance risk model was considered.

Theorem 3.2. *For $x > b$,*

$$V(x,b) = \frac{\alpha}{q} + \left(V(b,b) - \frac{\alpha}{q}\right) e^{-\Phi_1(q)(x-b)}. \tag{3.7}$$

Proof By using the strong Markov property of U_b at T_b^- as in Yin et al. (2013), we have

$$\begin{aligned} V(x,b) &= \frac{\alpha}{q} - \frac{\alpha}{q} E_x(e^{-qT_b^-}, T_b^- < \infty) \\ &\quad + E_x(e^{-qT_b^-} V(U_b(T_b^-), b), T_b^- < \infty). \end{aligned}$$

The result (3.7) follows since $P_x(U_b(T_b^-) = b) = 1$ and

$$E_x(e^{-qT_b^-}, T_b^- < \infty) = \exp(-\Phi_1(q)(x-b)).$$

Theorem 3.3. *For $0 < x < b$,*

$$\begin{aligned} V(x,b) &= -\frac{1}{2}\sigma^2 W^{(q)}(b-x) \\ &+ V(b,b)\Bigg(e^{\Phi_1(q)(b-x)} - \frac{1}{2}\sigma^2 \Phi_1(q) W^{(q)}(b-x) \\ &\quad + \alpha\Phi_1(q) e^{\Phi_1(q)(b-x)} \int_0^{b-x} W^{(q)}(z) e^{-\Phi_1(q)z} dz\Bigg) \\ &+ \frac{\alpha}{q}\left(\frac{1}{2}\sigma^2\Phi_1(q) W^{(q)}(b-x) - e^{\Phi_1(q)(b-x)}\Phi_1(q)\int_0^{b-x} Z^{(q)}(z) e^{-\Phi_1(q)z} dz\right) \\ &+ \frac{\alpha}{q} e^{\Phi_1(q)(b-x)}\left(q - \alpha\Phi_1(q)\right)\int_0^{b-x} W^{(q)}(z) e^{-\Phi_1(q)z} dz. \end{aligned} \tag{3.8}$$

where

$$V(b,b) = \frac{\frac{1}{2}\sigma^2 W^{(q)}(b) - B(b) - C(b)}{A(b)}.$$

Here

$$A(b) = e^{\Phi_1(q)(b)} - \frac{1}{2}\sigma^2\Phi_1(q) W^{(q)}(b) + \alpha\Phi_1(q) e^{\Phi_1(q)(b)} \int_0^b W^{(q)}(z) e^{-\Phi_1(q)z} dz,$$

$$B(b) = \frac{\alpha}{q}\left(\frac{1}{2}\sigma^2\Phi_1(q) W^{(q)}(b) - e^{\Phi_1(q)(b)}\Phi_1(q)\int_0^b Z^{(q)}(z) e^{-\Phi_1(q)z} dz\right),$$

$$C(b) = \frac{\alpha}{q} e^{\Phi_1(q)(b)}\left(q - \alpha\Phi_1(q)\right)\int_0^b W^{(q)}(z) e^{-\Phi_1(q)z} dz.$$

In particular, when $\sigma = 0$,

$$\begin{aligned} V(x,b) &= V(b,b)e^{\Phi_1(q)(b-x)}\left(1+\alpha\Phi_1(q)\int_0^{b-x}W^{(q)}(z)e^{-\Phi_1(q)z}dz\right) \\ &+ \frac{\alpha}{q}e^{\Phi_1(q)(b-x)}\left(q-\alpha\Phi_1(q)\right)\int_0^{b-x}W^{(q)}(z)e^{-\Phi_1(q)z}dz \\ &- \frac{\alpha}{q}e^{\Phi_1(q)(b-x)}\Phi_1(q)\int_0^{b-x}Z^{(q)}(z)e^{-\Phi_1(q)z}dz, \end{aligned} \tag{3.9}$$

where

$$V(b,b) = \frac{\frac{\alpha}{q}\Phi_1(q)\int_0^b Z^{(q)}(z)e^{-\Phi_1(q)z}dz - \frac{\alpha}{q}\left(q-\alpha\Phi_1(q)\right)\int_0^b W^{(q)}(z)e^{-\Phi_1(q)z}dz}{1+\alpha\Phi_1(q)\int_0^b W^{(q)}(z)e^{-\Phi_1(q)z}dz}.$$

Proof We first assume that $\lambda := \int_0^\infty \Pi(dy) < \infty$. Set $c_0 = c + \int_0^1 y\Pi(dy)$ and $f(y)dy = \frac{\Pi(dy)}{\lambda}$, then f is a probability density on $(0,\infty)$. In this case, in view of (3.7), the integro-differential equation (3.2) can be written as

$$\begin{aligned} \frac{1}{2}\sigma^2 V''(x,b) - c_0 V'(x,b) &= -\lambda\left[V(b,b)-\frac{\alpha}{q}\right]\int_{b-x}^\infty \exp(-\Phi_1(q)(x+y-b))f(y)dy \\ &\quad -\lambda\int_0^{b-x}V(x+y,b)f(y)dy - \frac{\lambda\alpha}{q}(1-F(b-x)) \\ &\quad +(\lambda+q)V(x,b),\ 0<x<b, \end{aligned} \tag{3.10}$$

where F is the distribution function of f. Replace the variable x by $z = b-x$, and define W by $W(z,b) = V(b-z,b), 0<z<b$. The (3.10) becomes

$$\begin{aligned} \frac{1}{2}\sigma^2 W''(z,b) + c_0 W'(z,b) &= -\lambda\left[W(0,b)-\frac{\alpha}{q}\right]\int_z^\infty \exp(-\Phi_1(q)(y-z))f(y)dy \\ &\quad -\lambda\int_0^{z}W(y,b))f(z-y)dy - \frac{\lambda\alpha}{q}(1-F(z)) \\ &\quad +(\lambda+q)W(z,b),\ 0<z<b, \end{aligned} \tag{3.11}$$

with initial condition $W(0,b) = V(b,b)$ and boundary condition $W(b,b) = 0$. We extend the definition of W by (3.11) to $z \geq 0$ and denote the resulting function by w. Then we have

$$\begin{aligned} \frac{1}{2}\sigma^2 w''(z) + c_0 w'(z) &= -\lambda\left[w(0)-\frac{\alpha}{q}\right]\int_z^\infty \exp(-\Phi_1(q)(y-z))f(y)dy \\ &\quad -\lambda\int_0^{z}w(y))f(z-y)dy - \frac{\lambda\alpha}{q}(1-F(z)) \\ &\quad +(\lambda+q)w(z),\ z\geq 0, \end{aligned} \tag{3.12}$$

with $w(0)=V(b,b)$ and $w(b)=0$.

For a function g, denoted by $\hat{g}$ the Laplace transform of g, i.e. $\hat{g}(\xi)=\int_0^\infty e^{-\xi y}g(y)dy$. Then the Laplace transform $\hat{w}$ for w can be easily determined from Eq. (3.11) as

$$\hat{w}(\xi)=\frac{\frac{1}{2}\sigma^2(-1+\xi w(0))+c_0w(0)+\frac{\lambda\alpha}{q\xi}(\hat{f}(\xi)-1)-\frac{\lambda}{\xi-\Phi_1(q)}(w(0)-\frac{\alpha}{q})(\hat{f}(\Phi_1(q))-\hat{f}(\xi)}{\frac{1}{2}\sigma^2\xi^2+c_0\xi-(\lambda+q)+\lambda\hat{f}(\xi)}. \tag{3.13}$$

Note that

$$\int_0^\infty e^{-x\xi}W^{(q)}(x)dx=\frac{1}{\frac{1}{2}\sigma^2\xi^2+c_0\xi-(\lambda+q)+\lambda\hat{f}(\xi)},$$

$$\int_0^\infty e^{-x\xi}dx\int_0^x W^{(q)}(y)dy=\frac{1}{\xi(\frac{1}{2}\sigma^2\xi^2+c_0\xi-(\lambda+q)+\lambda\hat{f}(\xi))},$$

$$\int_0^\infty e^{-x\xi}dx\int_0^x dy\int_0^y W^{(q)}(z)dz=\frac{1}{\xi^2(\frac{1}{2}\sigma^2\xi^2+c_0\xi-(\lambda+q)+\lambda\hat{f}(\xi))}.$$

Now inverting (3.13) gives

$$\begin{aligned} w(z) &= -\frac{1}{2}\sigma^2W^{(q)}(z)+\frac{1}{2}\sigma^2w(0)\int_0^z W^{(q)}(z-y)\delta_0'(y)dy \\ &\quad +c_0w(0)W^{(q)}(z)+\lambda w(0)\left(W^{(q)}*(f-\hat{f}(\Phi_1(q))\delta_0)*l\right)(z) \\ &\quad +\frac{\lambda\alpha}{q}\left(\Phi_1(q)\frac{Z^{(q)}-1}{q}*(\delta_0-f)*l\right)(z) \\ &\quad +\frac{\lambda\alpha}{q}\hat{f}(\Phi_1(q))-1)(W^{(q)}*l)(z), \end{aligned} \tag{3.14}$$

where δ_0 is the Dirac delta function at 0, h_1*h_2 stands for convolution of h_1 and h_2 and $l(z)=\exp(\Phi_1(q)z)$. After some tedious calculations, we get

$$\int_0^z W^{(q)}(z-y)\delta_0'(y)dy={W^{(q)}}'(z),$$

$$\begin{aligned} \lambda\left(W^{(q)}*(f-\hat{f}(\Phi_1(q))\delta_0)*l\right)(z) &= \alpha\Phi_1(q)\int_0^z W^{(q)}(z-y)e^{\Phi_1(q)y}dy \\ &\quad +e^{\Phi_1(q)z}-\frac{1}{2}\sigma^2{W^{(q)}}'(z) \\ &\quad -(\frac{1}{2}\sigma^2\Phi_1(q)+c_0)W^{(q)}(z), \end{aligned}$$

$$\begin{aligned} \lambda(Z^{(q)}-1)*(\delta_0-f)*l)(z) &= \left(\frac{1}{2}\sigma^2q\Phi_1(q)+c_0q\right)\int_0^z W^{(q)}(z-y)e^{\Phi_1(q)y}dy \\ &\quad +\frac{1}{2}\sigma^2qW^{(q)}(z)-q\int_0^z Z^{(q)}(z-y)e^{\Phi_1(q)y}dy. \end{aligned}$$

Substituting the three expressions above into (3.14) we arrive at

$$\begin{aligned}
w(z) &= -\frac{1}{2}\sigma^2 W^{(q)}(z) \\
&+ w(0)\Bigg(e^{\Phi_1(q)z} - \frac{1}{2}\sigma^2\Phi_1(q)W^{(q)}(z) \\
&\quad +\alpha\Phi_1(q)e^{\Phi_1(q)z}\int_0^z W^{(q)}(y)e^{-\Phi_1(q)y}dy\Bigg) \\
&+ \frac{\alpha}{q}\left(\frac{1}{2}\sigma^2\Phi_1(q)W^{(q)}z - e^{\Phi_1(q)z}\Phi_1(q)\int_0^z Z^{(q)}(y)e^{-\Phi_1(q)y}dy\right) \\
&+ \frac{\alpha}{q}e^{\Phi_1(q)z}\left(q-\alpha\Phi_1(q)\right)\int_0^z W^{(q)}(y)e^{-\Phi_1(q)y}dy, \qquad (3.15)
\end{aligned}$$

and the result (3.8) follows since $V(x,b) = w(b-x)$ and $w(0) = V(b,b)$.

Now, we assume that $\lambda := \int_0^\infty \Pi(dy) = \infty$. Let Π_n be measures on $(0,\infty)$ defined by

$$\Pi_n(dx) = \Pi(dx)\mathbf{1}_{\{(\frac{1}{n},\infty)\}}(x),\ n \geq 1.$$

Then we have

$$\lambda_n := \int_0^\infty \Pi_n(dx) \leq n^2\int_{\frac{1}{n}}^1 x^2\Pi(dx) + \int_1^\infty (1\wedge x^2)\Pi(dx) < \infty.$$

Set $c_n = c + \int_0^1 y\Pi_n(dy)$ and $f_n(y)dy = \frac{\Pi_n(dy)}{\lambda_n}$, then for each $n \geq 1$, f_n is a probability density on $(0,\infty)$. Similar to (3.10) we consider the following integro-differential equation

$$\begin{aligned}
\frac{1}{2}\sigma^2 {V_n}''(x,b) - c_n{V_n}'(x,b) &= -\lambda\left[V_n(b,b) - \frac{\alpha}{q}\right]\int_{b-x}^\infty \exp(-\Theta_n(q)(x+y-b))f_n(y)dy \\
&\quad -\lambda\int_0^{b-x} V_n(x+y,b)f_n(y)dy - \frac{\lambda\alpha}{q}(1-F_n(b-x)) \\
&\quad +(\lambda+q)V_n(x,b),\ 0<x<b, \qquad (3.16)
\end{aligned}$$

where $\Theta_n(q) = \sup\{\theta \geq 0 | \Psi_n(\theta) + \alpha\theta = q\}$. Here,

$$\Psi_n(\theta) = c_n\theta + \frac{1}{2}\sigma^2\theta^2 + \int_0^\infty (e^{-\theta x} - 1 + \theta x\mathbf{1}_{\{|x|<1\}})\Pi_n(dx). \qquad (3.17)$$

Repeating the same argument as the case that $\lambda := \int_0^\infty \Pi(dy) < \infty$, we obtain

$$
\begin{aligned}
V_n(x,b) &= -\frac{1}{2}\sigma^2 W_n^{(q)}(b-x) \\
&+ V_n(b,b)\Bigg(e^{\Theta_n(q)(b-x)} - \frac{1}{2}\sigma^2\Theta_n(q)W_n^{(q)}(b-x) \\
&\quad +\alpha\Theta_n(q)e^{\Theta_n(q)(b-x)}\int_0^{b-x} W_n^{(q)}(z)e^{-\Theta_n(q)z}dz\Bigg) \\
&+ \frac{\alpha}{q}\left(\frac{1}{2}\sigma^2\Theta_n(q)W_n^{(q)}(b-x) - e^{\Theta_n(q)(b-x)}\Theta_n(q)\int_0^{b-x} Z_n^{(q)}(z)e^{-\Theta_n(q)z}dz\right) \\
&+ \frac{\alpha}{q}e^{\Theta_n(q)(b-x)}\left(q-\alpha\Theta_n(q)\right)\int_0^{b-x} W_n^{(q)}(z)e^{-\Theta_n(q)z}dz, \qquad (3.18)
\end{aligned}
$$

where $W_n^{(q)}$ and $Z_n^{(q)}$ are scale functions corresponding to the process X_n with Laplace exponent Ψ_n. Since $\lim_{n\to\infty}\Psi_n(\theta) = \Psi(\theta)$, then $X_n \to X$ weakly as $n \to \infty$. Thus $\lim_{n\to\infty} V_n = V$, $\lim_{n\to\infty} W_n^{(q)} = W^{(q)}$, $\lim_{n\to\infty} Z_n^{(q)} = Z^{(q)}$ and $\lim_{n\to\infty}\Theta_n = \Phi_1$. Consequently, (3.8) still holds for this case. This ends the proof of Theorem 3.3.

Remark 3.2. *For $\sigma \geq 0$, from the graph of*

$$\Psi(\Phi_1(q)) + \alpha\Phi_1(q) = q$$

one can verify that $\Phi_1(q) \to 0$ when $\alpha \to \infty$. After some simple calculations we get $\lim_{\alpha\to\infty}\alpha\Phi_1(q) = q$,

$$\lim_{\alpha\to\infty}\alpha(q-\alpha\Phi_1(q)) = q\Psi'(0+),$$

and

$$\lim_{\alpha\to\infty} V(b,b) = \frac{\overline{Z}^{(q)}(b)}{Z^{(q)}(b)} + \frac{\Psi'(0+)}{qZ^{(q)}(b)} - \frac{\Psi'(0+)}{q}.$$

As a result, for $0 < x < b$, we arrive at

$$\lim_{\alpha\to\infty} V(x,b) = \frac{\overline{Z}^{(q)}(b)}{Z^{(q)}(b)}Z^{(q)}(b-x) - \overline{Z}^{(q)}(b-x) + \frac{\Psi'(0+)}{q}\left(\frac{Z^{(q)}(b-x)}{Z^{(q)}(b)} - 1\right),$$

which is the expected discounted value of dividend payments for the barrier strategy. See Lemma 2.1 of Bayraktar, Kyprianou and Yamazaki (2013).

4 Optimal dividend strategy

Suppose the maximal dividend-value function V is $C^2(0,\infty)$ (resp. $C^1(0,\infty)$) when X is of unbounded (resp. bounded) variation. Standard Markovian arguments, see Fleming

and Soner(1993), formally yield that $V(x)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation

$$\Gamma V(x) - qV(x) + \sup_{0\le r\le\alpha}\{r(1-V'(x))\} = 0 \tag{4.1}$$

with $V(0)=0$, where Γ is the extended generator of the process X, which defined by

$$\Gamma g(x) = \frac{1}{2}\sigma^2 g''(x) - cg'(x) + \int_0^\infty [g(x+y) - g(x) - g'(x)y\mathbf{1}_{\{|y|<1\}}]\Pi(dy). \tag{4.2}$$

It follows from the HJB equation (4.1) that an optimal dividend policy has to fulfil $r=0$ if $V'(U^\pi_{t-}) > 1$; $r=\alpha$ if $V'(U^\pi_{t-}) < 1$. In some situations the optimal dividend strategy is a threshold strategy. In fact if $V'(x,0) < 1$ for $x>0$, then the threshold strategy with $b^*=0$ is optimal, if $V'(x,b^*) > 1$ for $x<b^*$ and $V'(x,b^*) < 1$ for $x>b^*$, then the threshold strategy with $b^*>0$ is optimal. The optimal threshold b^* can be obtained by $V'(b^*,b^*)=1$. From those facts one sees that if $V(x,b^*)$ is a concave function on $(0,\infty)$, then the optimal dividend strategy is a threshold strategy.

It follows from (3.7) that

$$V'(x,b) = -\Phi_1(q)\left(V(b,b) - \frac{\alpha}{q}\right)e^{-\Phi_1(q)(x-b)},\ x>b. \tag{4.3}$$

If $\Phi_1(q)\frac{\alpha}{q} \le 1$, then $V'(x,0)\le 1$ since $V(0,0)=0$. Thus $b^*=0$. Now, suppose that $\Phi_1(q)\frac{\alpha}{q} > 1$, we get the condition for b^*:

$$-\Phi_1(q)\left(V(b^*,b^*) - \frac{\alpha}{q}\right) = 1.$$

Or, equivalently, b^* is the solution of the equation

$$V(b^*,b^*) = \frac{\alpha}{q} - \frac{1}{\Phi_1(q)}. \tag{4.4}$$

From (3.7) and (3.8) we get

$$V(x,b^*) = \begin{cases} \frac{\alpha}{q} - \frac{1}{\Phi_1(q)}e^{-\Phi_1(q)(x-b^*)}, & \text{if } x>b^*,\\ -\frac{\alpha\int_0^{b^*-x}W^{(q)}(z)e^{-\Phi_1(q)z}dz}{e^{-\Phi_1(q)(b^*-x)}} + \frac{\alpha}{q}Z^{(q)}(b^*-x) - \frac{e^{\Phi_1(q)(b^*-x)}}{\Phi_1(q)}, & \text{if } 0<x<b^*. \end{cases} \tag{4.5}$$

Taking derivative with respect to x in the both sides of the above relation leads to

$$V'(x,b^*) = \begin{cases} e^{-\Phi_1(q)(x-b^*)}, & \text{if } x>b^*,\\ e^{\Phi_1(q)(b^*-x)}\left(1+\alpha\Phi_1(q)\int_0^{b^*-x}W^{(q)}(z)e^{-\Phi_1(q)z}dz\right), & \text{if } 0<x<b^*. \end{cases} \tag{4.6}$$

It follows that $V'(x,b^*)<1$ when $x>b^*$. Further, for $0<x<b^*$,

$$\begin{aligned} V''(x,b^*) &= -\Phi_1(q)e^{\Phi_1(q)(b^*-x)}\left(1+\alpha\Phi_1(q)\int_0^{b^*-x}W^{(q)}(z)e^{-\Phi_1(q)z}dz\right) \\ &- \alpha\Phi_1(q)W^{(q)}(b^*-x)<0. \end{aligned} \tag{4.7}$$

Thus for $0<x<b^*$, $V'(x,b^*)>V'(b^*,b^*)=1$.

Remark 4.1. *From (4.6) and (4.7) we find that $V'(x,b^*)$ is continuous on $(0,\infty)$, $V''(b^*-,b^*) = -\Phi_1(q)-\alpha\Phi_1(q)W^{(q)}(0)$ and $V''(b^*+,b^*) = -\Phi_1(q)$. So that if $\sigma=0$, then $V''(b^*-,b^*)\neq V''(b^*+,b^*)$ and, if $\sigma\neq 0$, then $V''(b^*-,b^*)=V''(b^*+,b^*)$.*

In summary, we have the following:

Theorem 4.1. *For any spectrally positive Lévy process (2.3), consider the stochastic control problem (2.8). Let Ξ be the class of admissible dividend strategies satisfying (2.5)-(2.7). Then we have $V(x)=V(x,b^*)$ as defined in (4.5) and the threshold strategy with threshold b^* is the optimal dividend strategy over Ξ.*

Remark 4.2. *It is interesting to note that the optimal strategy with bounded rate of dividend payment is formed by a threshold strategy for spectrally positive Lévy process regardless of the Lévy measure. However, for the spectrally negative Lévy model $-X$, it was shown in Kyprianou, Loeffen and Pérez (2012) that the optimal strategy is formed by a threshold strategy under certain regularity condition on the Lévy measure of $-X$.*

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